META – TEOP
Graph Theory Workshop
Dr. Stanley R. Huddy

Mathematics Enrichment Through Applications
Technical Enrichment and Outreach Program
Sponsored by Time Warner Cable

FAIRLEIGH DICKINSON UNIVERSITY

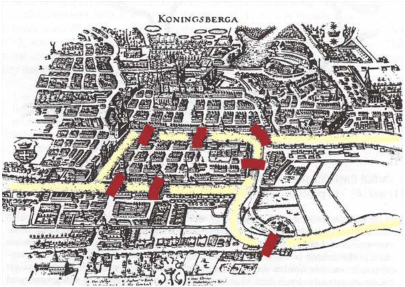
META-TEOP Graph Theory Workshop

What we will cover in this workshop:

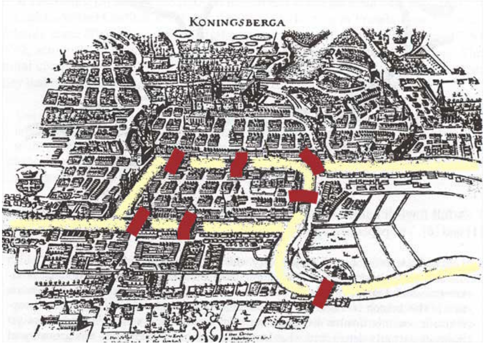
- Basic Graph Theory
  - Graphs
  - Eulerian and Hamiltonian graphs
  - Weighted Graphs
  - Directed Graphs
  - Subgraphs
  - Regular Graphs
  - Planar Graphs
  - Trees
- Applications of Graph Theory
  - The Bridges of Königsberg
  - The Traveling Salesman
  - GPS – Global Positioning System
  - Instant Insanity
  - Three Houses and Three Utilities Problem

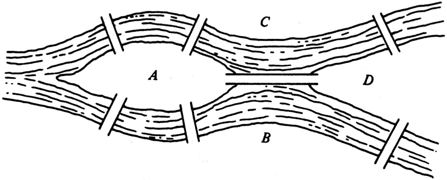
The Bridges of Königsberg

In the town of Königsberg there were in the 18th century seven bridges witch crossed the river Pregel. They connected two islands in the river with each other and with opposite banks. The townsfolk had long amused themselves with this problem: Is it possible to cross the seven bridges in a continuous walk without recrossing any of them?

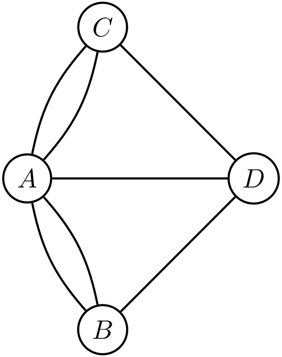


The Bridges of Königsberg





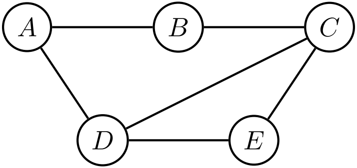
Is the graph **Eulerian**?



What Are Graphs?

A **graph** is an ordered pair $G = (V, E)$, where V is a set of **vertices** (or nodes) and E is a set of **edges**.

For example,



where,

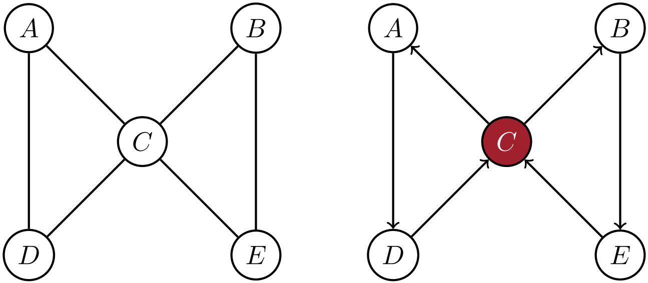
$$\text{Vertices:} \quad V = \{A, B, C, D, E\}$$

$$\text{Edges:} \quad E = \{AB, AD, BC, CD, CE, DE\}$$

Eulerian Graphs

A graph is called **Eulerian** if it contains an **Eulerian cycle**.

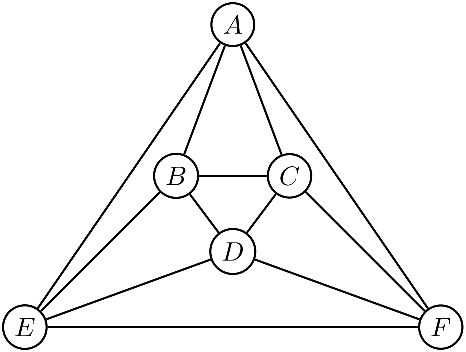
An **Eulerian cycle** visits every edge exactly once and returns to its starting node.



Eulerian cycle: CA, AD, DC, CB, BE, EC

Challenge Problem 1

Does the graph have an **Eulerian cycle**?



Eulerian cycle: $CD, DB, BA, AE, EF, FA, AC, CF, FD, DE, EB, BC$

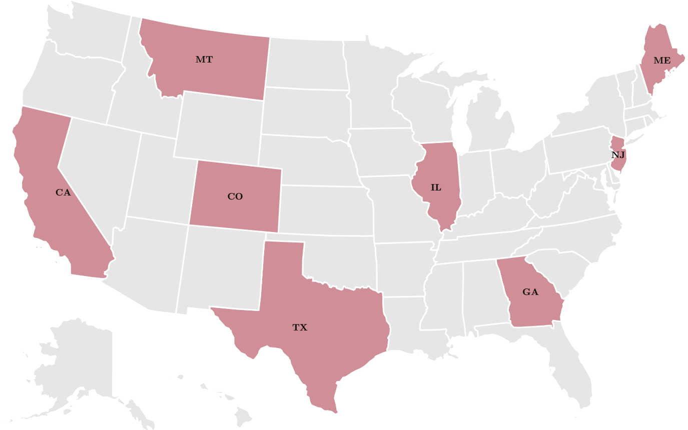
Eulerian Graphs

The bridges of Königsberg problem was initially solved by the famous Swiss mathematician Leonhard Euler (1707 – 1783).

A graph is Eulerian if every vertex of a graph has even degree.

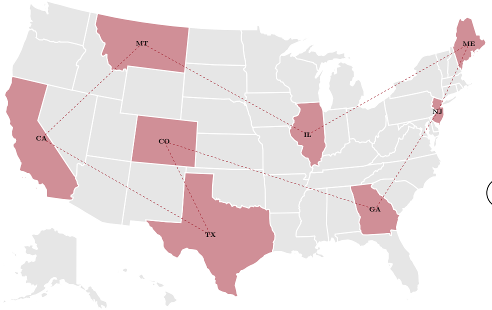
The Traveling Salesman

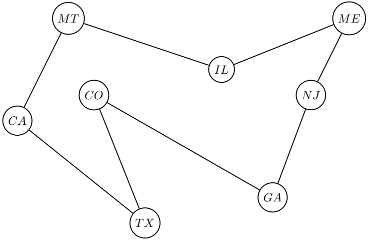
A traveling salesman wishes to visit several cities and return home to his starting point.



The Traveling Salesman

We can represent the salesman's travels as a graph whose vertices correspond to the cities he wishes to visit.



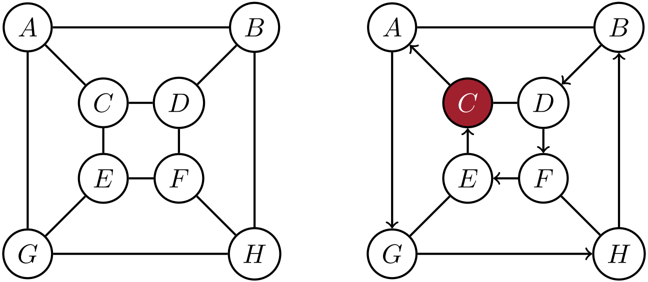


Is the graph **Hamiltonian**?

Hamiltonian Graphs

A graph is called **Hamiltonian** if it contains a **Hamiltonian cycle**.

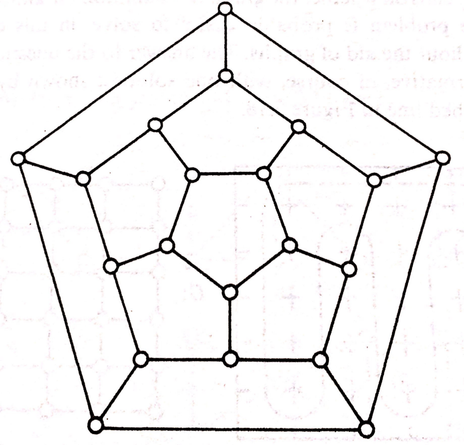
A **Hamiltonian cycle** visits every node exactly once and returns to its starting node.



Hamiltonian cycle: $CA, AG, GH, HB, BD, DF, FE, EC$

Challenge Problem 2

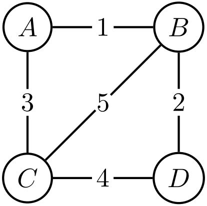
Does the graph have a **Hamiltonian cycle**?



Weighted Graphs

A **weighted graph** has a number assigned to each edge.

For example,

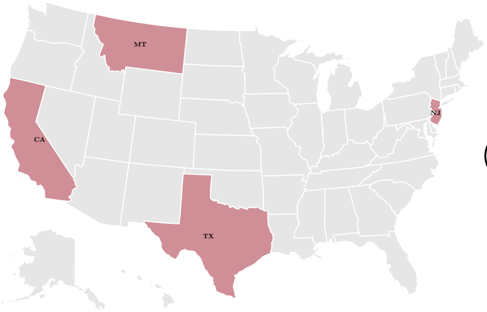


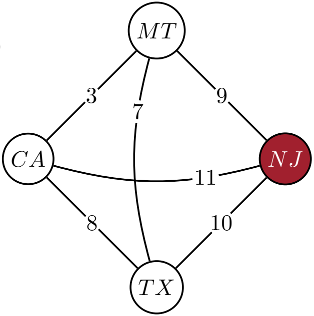
Therefore, each Hamiltonian cycle (or Eulerian cycle) can be associated with a number.

Hamiltonian cycle: $AB, BD, DC, CA = 1 + 2 + 4 + 3 = 10.$

The Traveling Salesman

To find which Hamiltonian cycle is the optimal travel route for our salesman, we can use a weighted graph to represent the "cost" of him traveling from one city to another city.

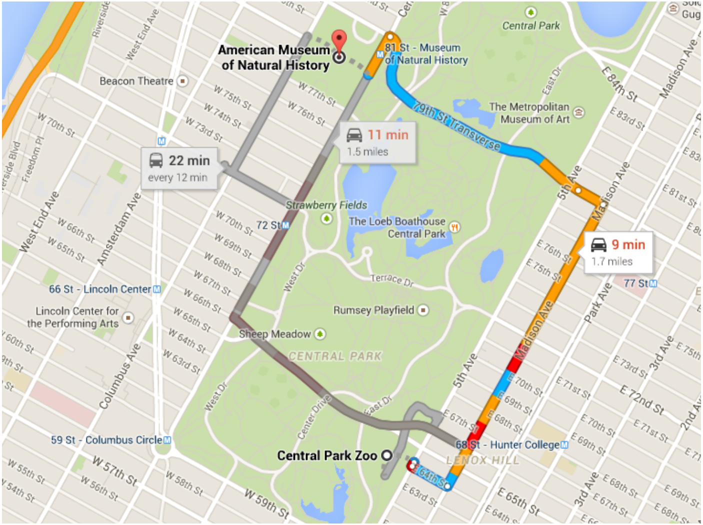




Optimal Hamiltonian cycle:

$NJ - MT, MT - CA, CA - TX, TX - NJ = 9 + 3 + 8 + 10 = 30$

GPS

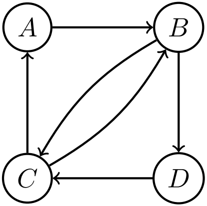

American Museum of Natural History
81 St - Museum of Natural History
Central Park
Beacon Theatre
79th St Transverse
The Metropolitan Museum of Art
11 min
1.5 miles
22 min
every 12 min
Strawberry Fields
72 St
The Loeb Boathouse Central Park
9 min
1.7 miles
77 St
66 St - Lincoln Center
Lincoln Center for the Performing Arts
Rumsey Playfield
Sheep Meadow
CENTRAL PARK
59 St - Columbus Circle
Central Park Zoo
68 St - Hunter College
LENOX HILL
W 57th St
W 59th St
E 65th St
E 72nd St
Madison Ave
5th Ave
Park Ave
Columbus Ave
Amsterdam Ave
West End Ave
E 84th St

Directed Graphs

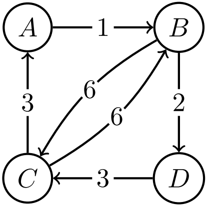
A **directed graph** has directed edges. The edges of a directed graph can also be weighted.

For example,

G_1 :



G_2 :



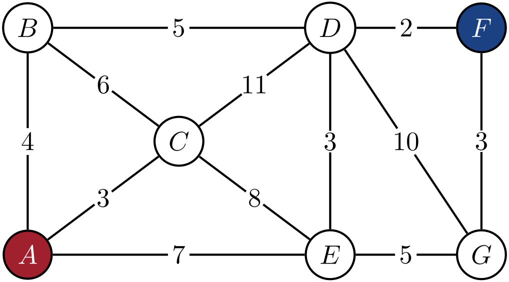
What is the **shortest path** from node A to node C?

For graph G_1: AB, BC
For graph G_2: $AB, BD, DC = 1 + 2 + 3 = 6$

Dijkstra's Algorithm

1. Assign to every node a tentative distance value: set it to zero for our initial node and to infinity for all other nodes.
2. Set the initial node as current. Mark all other nodes unvisited. Create a set of all the unvisited nodes called the unvisited set.
3. For the current node, consider all of its unvisited neighbors and calculate their tentative distances. Compare the newly calculated tentative distance to the current assigned value and assign the smaller one.
4. When we are done considering all of the neighbors of the current node, mark the current node as visited and remove it from the unvisited set. A visited node will never be checked again.
5. Select the unvisited node that is marked with the smallest tentative distance, and set it as the new "current node" then go back to step 3.
6. If the destination node has been marked visited, then stop. The algorithm has finished.

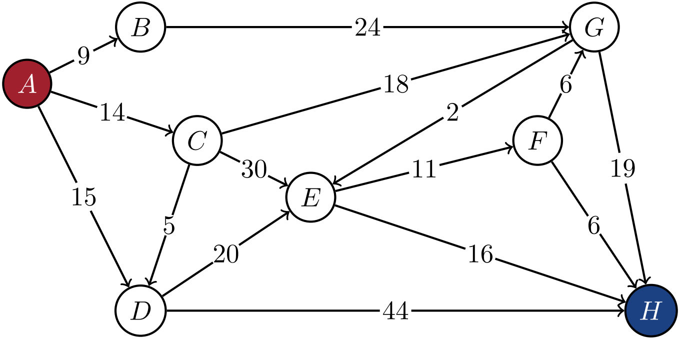
Dijkstra's Algorithm



Unvisited set $= \{A, B, C, D, E, F, G\}$.
Shortest path $= AB, BD, DF = 4 + 5 + 2 = 9$.

Challenge Problem 3

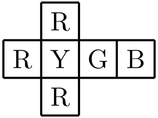
Find the **shortest path** from node A to node H.



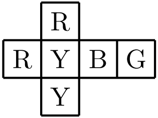
Instant Insanity

Given four cubes whose faces are colored red, blue, green, and yellow, can we pile them up so that all four colors appear on each side of the resulting stack?

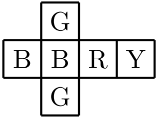
Cube 1:



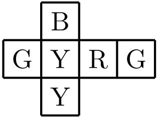
Cube 2:

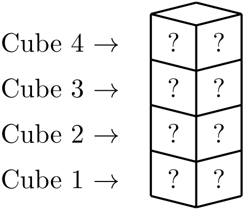


Cube 3:



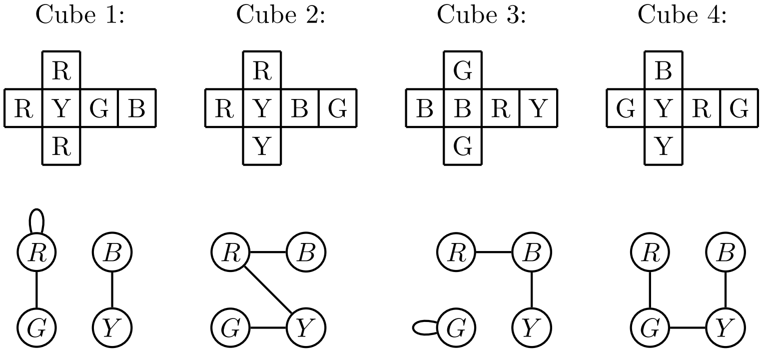
Cube 4:





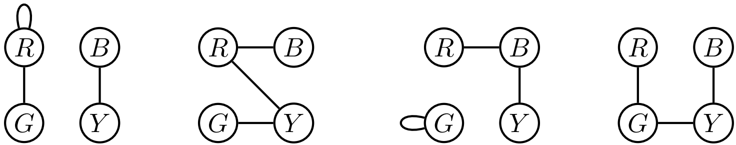
Instant Insanity

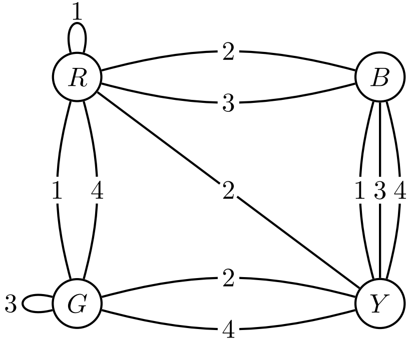
We represent each cube by a graph with four vertices. We draw an edge between two vertices if the cube has corresponding colors on opposite faces.



Instant Insanity

We then superimpose these graphs to form a new graph.





Instant Insanity

We then need to find two subgraphs of the superimposed graph. The first subgraph tells us which pair of colors appears on the front and back of each cube, and the second subgraph tells us which pair of colors appears on the left and right of each cube.

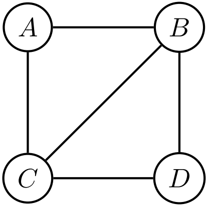
Subgraph properties:

1. Each subgraph contains exactly one edge from each cube.
2. The subgraphs have to edges in common.
3. Each subgraph is regular of degree 2.

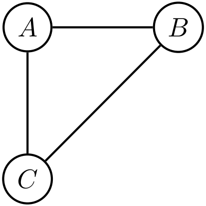
Subgraphs

A **subgraph** is a "piece" of a graph.

For the graph:



A subgraph is:



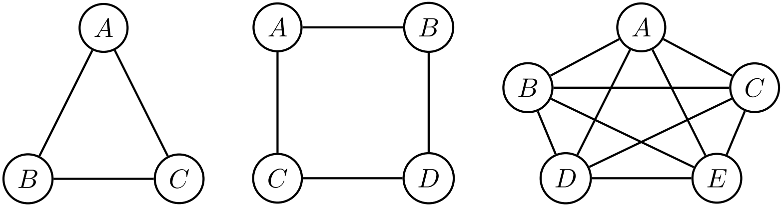
Can you draw another subgraph?

Regular Graphs

The nodes of a **regular graph** all have the same vertex degree.

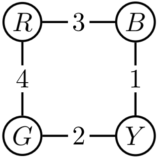
The **vertex degree** is the number of edges connected to that vertex.

For example,

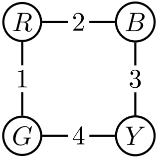


Instant Insanity

The two subgraphs are:

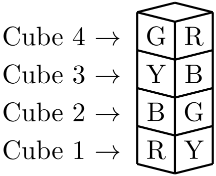


Front and Back

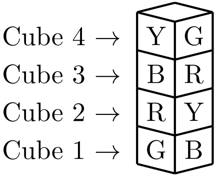


Left and Right

Therefore, the answer is:



Left and Front

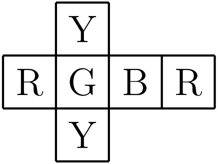


Right and Back

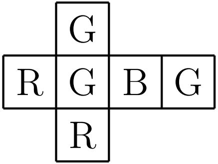
Challenge Problem 4

Find a solution to the Instant Insanity problem.

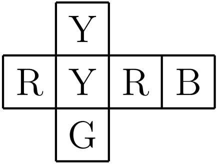
Cube 1:



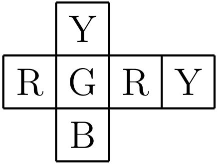
Cube 2:



Cube 3:

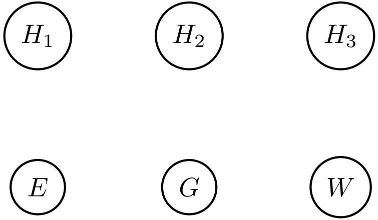


Cube 4:



The Three Houses and Three Utilities Problem

Suppose we have three houses and three utility outlets (electric, gas, and water) situated as shown below. Is it possible to connect each utility with each of the three houses without the lines or mains crossing?

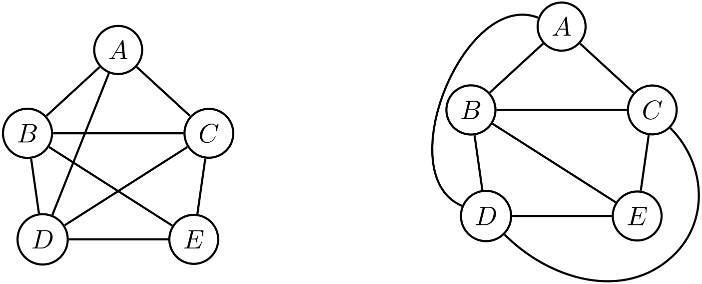


Can we create a planar graph using the nodes above?

Planar Graphs

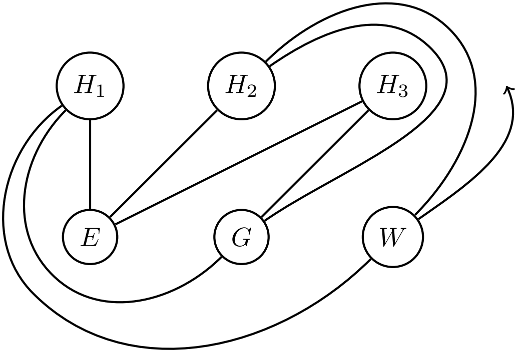
A planar graph is a graph that can be drawn in the plane in such a way that no two edges intersect except at a vertex.

The graph on the left is a planar graph because it can be redrawn as the graph on the right.



The Three Houses and Three Utilities Problem

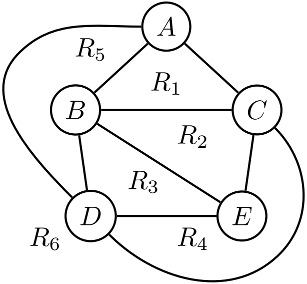
The graph is not a planar graph.



Therefore, it is not possible to connect each utility with each of the three houses without the lines or mains crossing.

Planar Graphs and Induction

A planar graph separates the plane into regions.



Euler's Formula:

If a planer graph has v vertices, e edges, and r regions, then

$$v - e + r = 2.$$

Proof by Induction

The simplest and most common form of mathematical induction infers that a statement involving a natural number n holds for all values of n.

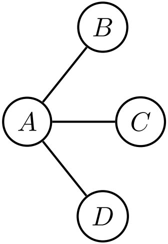
The proof consists of two steps:

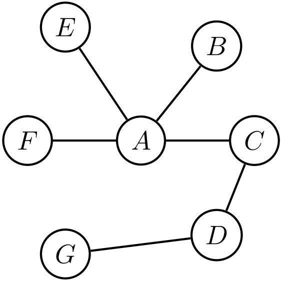
1. The basis (base case): prove that the statement holds for the first natural number n. Usually, $n = 0$ or $n = 1$.
2. The inductive step: prove that, if the statement holds for some natural number n, then the statement holds for $n + 1$.

Trees

A tree is a connected graph which has no cycles.

For example,





Proving Euler's Formula by Induction

Euler's Formula:
If a planer graph G has v vertices, e edges, and r regions, then

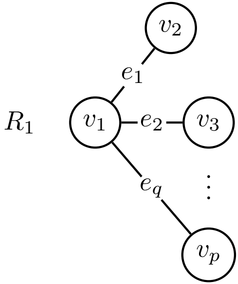
$$v - e + r = 2.$$

Proof:
Base case: If the number of edges $e = 0$, then the number of vertices must be $v = 1$ since G is connected, and the number of regions is $r = 1$.

Therefore, $v - e + r = 1 - 0 + 1 = 2$, and the formula is true for the base case.

Proving Euler's Formula by Induction

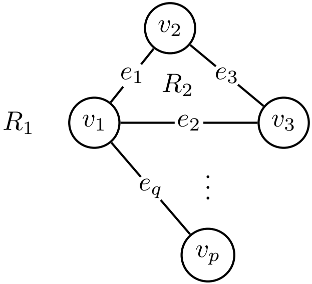
Inductive step: Suppose that the formula holds for all graphs with at most $e-1$ edges and let G be a graph with e edges. If G is a tree, then $v = e+1$ and $r = 1$.



Thus $v - e + r = (e+1) - e + 1 = 2$ as required.

Proving Euler's Formula by Induction

If G is not a tree.



Let e_c be an edge in some cycle of G. Then $G - e_c$ is a connected planar graph with v vertices, $e - 1$ edges, and $r - 1$ regions. By the induction hypothesis, $v - (e - 1) + (r - 1) = 2$ and therefore $v - e + r = 2$, as required.